Pion Number Fluctuations and Correlations in the Statistical System with Fixed Isospin

V. V. Begun,[1,2] M. I. Gorenstein,[1,2] and O. A. Mogilevsky[1]

[1]*Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine*

[2]*Frankfurt Institute for Advanced Studies, Frankfurt, Germany*

Abstract

The statistical system of pions with zero total isospin is studied. The suppression effects for the average yields due to isospin conservation are the same for π^0, π^+ and π^-. However, a behavior of the corresponding particle number fluctuations are different. For neutral pions there is the enhancement of the fluctuations, whereas for charged pions the isospin conservation suppresses fluctuations. The correlations between the numbers of charged and neutral pions are observed for finite systems. This causes a maximum of the total pion number fluctuations for small systems. The thermodynamic limit values for the scaled variances of neutral and charged pions are calculated. The enhancements of the fluctuations due to Bose statistics are found and discussed.

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I. INTRODUCTION

The aim of the present paper is to study some aspects of non-Abelian symmetries in the statistical models. We consider SU(2)-isospin symmetry group for the pion system. The role of the isospin conservation in a many body system was first considered in the pioneering paper of Bethe [1]. Many efforts were then aimed at studies of the pion system with fixed isospin [2, 3]. An effective theoretical formalism for non-Abelian symmetries in the statistical mechanics was developed in Ref. [4] on the basis of the group projection technique. It allowed to consider the impact of the isospin conservation on the particle abundances and the form of their momentum spectra in the statistical models of hadron production [5, 6]. The group projection technique was also used to calculate the colorless partition function of the quark-gluon gas with SU(N_c)-color symmetry [7, 8].

Our primary interest in the present paper is to study an influence of non-Abelian charge conservation on the particle number fluctuations. It was recently found [9] that exact conservation of Abelian (additive) charges causes the suppression of the particle number fluctuations. In the present study we restrict our consideration to the simplest statistical system with non-Abelian symmetry – an ideal pion gas with zero isospin $I = 0$. Most discussions are done within Boltzmann statistics. This makes possible to obtain transparent analytical results and compare them with those in the canonical ensemble for zero electric charge $Q = 0$.

The paper is organized as the following. In Section II we consider the partition function and total number of pions (the mean value and scaled variance). In Section III we calculate the mean multiplicities, fluctuations and correlations for π^+, π^- and π^0 mesons. In Section IV we consider the specific effects due to Bose statistics. The Section V summarizes the paper.

II. PARTITION FUNCTION AND TOTAL NUMBER OF PIONS

The partition function of the ideal Boltzmann gas of pions $\pi^+,\ \pi^-,\ \pi^0$ in the grand canonical ensemble (GCE) reads,

$$Z_{gce} \;=\; \sum_{N_0,N_+,N_-=0}^{\infty} \frac{(\lambda_0\ z)^{N_0}}{N_0!}\ \frac{(\lambda_+\ z)^{N_+}}{N_+!}\ \frac{(\lambda_-\ z)^{N_-}}{N_-!} \;=\; \exp\left[(\lambda_0+\lambda_++\lambda_-)\,z\right]\ , \tag{1}$$

where z is the one-particle partition function,

$$z \;=\; \frac{V}{2\pi^2} \int_0^\infty p^2 dp\; \exp\left(-\;\frac{\sqrt{p^2+m^2}}{T} \right) \;=\; \frac{V}{2\pi^2}\; Tm^2\; K_2\left(\frac{m}{T}\right)\;. \tag{2}$$

Here V and T are the system volume and temperature, m is the pion mass (we neglect a small difference between the masses of charged and neutral pions), and K_2 is the modified Hankel function. The auxiliary parameters λ_j with $j = 0,\ +,\ -$ are introduced to calculate the mean pion multiplicities, fluctuations and correlations. We take $\lambda_j \equiv 1$ in the final formulae.

In the case of exact charge conservation, i.e. in the canonical ensemble (CE) with zero charge $Q = 0$, the partition function is (see, e.g. [9, 10]):

$$\begin{aligned} Z_{Q=0} \;&=\; \sum_{N_0,N_+,N_-=0}^{\infty} \delta\left(N_+ \;-\; N_-\right)\; \frac{(\lambda_0\; z)^{N_0}}{N_0!}\; \frac{(\lambda_+\; z)^{N_+}}{N_+!}\; \frac{(\lambda_-\; z)^{N_-}}{N_-!} \\ &=\; \exp\left(\lambda_0\; z\right)\; \frac{1}{2\pi} \int_0^{2\pi} d\phi\; \exp\left[\; z\;\; (\lambda_+ \exp[i\phi] \;+\; \lambda_- \exp[-i\phi])\;\right]\;. \end{aligned} \tag{3}$$

Note that an exact charge conservation in the CE (3) does not affect the neutral pions. Their number distribution remains the Poissonian one, the same as in the GCE (1).

The partition function with total isospin $I = 0$ can be obtained using group projection technique. Pions are transformed under vector (adjoint) representation of the SU(2) group. This group has three parameters which can be chosen as Euler angles $\vec{\alpha} = \alpha, \beta, \gamma$. In this case the diagonal matrix elements have the following form [11]:

$$D^1_{\pm1,\pm1}(\alpha,\beta,\gamma) \;=\; e^{\pm i(\alpha+\gamma)} \left(\frac{1+\cos(\beta)}{2}\right)\;, \qquad D^1_{0,0}(\alpha,\beta,\gamma) \;=\; \cos(\beta)\;. \tag{4}$$

The partition function is then presented as [6, 12]:

$$\begin{aligned} Z_{I=0} \;&=\; \int d\mu \sum_{N_0,N_+,N_-=0}^{\infty} \frac{\left[\lambda_0\; z\; D^1_{0,0}(\vec{\alpha})\right]^{N_0}}{N_0!}\; \frac{\left[\lambda_+\; z\; D^1_{1,1}(\vec{\alpha})\right]^{N_+}}{N_+!}\; \frac{\left[\lambda_-\; z\; D^1_{-1,-1}(\vec{\alpha})\right]^{N_-}}{N_-!} \\ &=\; \int d\mu\; \exp\left[\lambda_0\; z\; D^1_{0,0}(\vec{\alpha}) \;+\; \lambda_+\; z\; D^1_{1,1}(\vec{\alpha}) \;+\; \lambda_-\; z\; D^1_{-1,-1}(\vec{\alpha})\right]\;. \end{aligned} \tag{5}$$

Substituting explicit expressions for the Haar group measure $d\mu$ and matrix elements $D^t_{t_3,t_3}$ (4) in Eq. (5), one obtains:

$$\begin{aligned} Z_{I=0} \;&=\; \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta \\ &\times\; \exp\left[\lambda_0\, z\, \cos\beta \;+\; z\left(\frac{1+\cos\beta}{2}\right)\; \left(\lambda_+\; \exp[i(\alpha+\gamma)] \;+\; \lambda_-\; \exp[-i(\alpha+\gamma)]\right)\right]\;. \end{aligned} \tag{6}$$

The change of variables $\phi = \alpha + \gamma$, $\varphi = (\alpha - \gamma)/2$, $\cos(\beta) = x$ and integration over φ gives:

$$Z_{I=0} \;=\; \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} dx \exp\left[\lambda_0\, z\, x \;+\; z\,\frac{1+x}{2}\,(\lambda_+ \exp[i\phi] \;+\; \lambda_- \exp[-i\phi])\right] . \tag{7}$$

Comparing $Z_{I=0}$ (7) with the partition function $Z_{Q=0}$ (3) one observes an additional x-integration in Eq. (7). It reflects a presence of the particle number correlations between neutral and charged pions which were absent in the GCE and CE.

The partition functions (1), (3), and (7) can be simplified by taking $\lambda_j = \lambda$. One finds:

$$Z_{gce} = \exp(3\lambda z)\ , \quad Z_{Q=0} = \exp(\lambda z)\ I_0(2\lambda z)\ , \quad Z_{I=0} = \exp(\lambda z)\ [I_0(2\lambda z) \;-\; I_1(2\lambda z)]\ , \tag{8}$$

where I_n are the modified Bessel functions. The final expressions for the partition functions correspond to $\lambda = 1$. Taking the derivatives of Z over T and V one finds the thermodynamical functions of the pion system. The derivatives over λ give the moments of total pion number distribution,

$$\langle N\rangle \;=\; \frac{1}{Z}\frac{\partial Z}{\partial\lambda}\bigg|_{\lambda=1}\ , \qquad \langle N^2\rangle \;=\; \frac{1}{Z}\frac{\partial}{\partial\lambda}\left(\lambda\frac{\partial Z}{\partial\lambda}\right)\bigg|_{\lambda=1} . \tag{9}$$

With Eq. (9) one calculates the average pion multiplicity $\langle N\rangle$ and the second moment $\langle N^2\rangle$ using the partition functions (8) of different statistical ensembles:

$$\langle N\rangle_{gce} \;=\; 3z\ , \qquad \langle N^2\rangle_{gce} \;=\; 3z \;+\; 9z^2\ , \tag{10}$$

$$\langle N\rangle_{Q=0} \;=\; z \;+\; 2z\,\frac{I_1(2z)}{I_0(2z)}\ , \qquad \langle N^2\rangle_{Q=0} \;=\; z \;+\; 5z^2 \;+\; 4z^2\,\frac{I_1(2z)}{I_0(2z)}\ , \tag{11}$$

$$\langle N\rangle_{I=0} \;=\; z\,\frac{I_1(2z) - I_2(2z)}{I_0(2z) - I_1(2z)}\ , \qquad \langle N^2\rangle_{I=0} \;=\; \langle N\rangle_{I=0} \;+\; z^2\,\frac{I_0(2z) \;-\; I_3(2z)}{I_0(2z) - I_1(2z)} . \tag{12}$$

The ratio R_N and the scaled variance ω_N,

$$R_N \;\equiv\; \frac{\langle N\rangle}{3z}\ , \qquad \omega_N \;\equiv\; \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle}\ , \tag{13}$$

for the total number of pions calculated in different statistical ensembles are shown in Fig. 1. In the GCE one obtains $R_N(GCE) = \omega_N(GCE) = 1$.

Figure 1 shows that conservation laws suppress the mean particle number. For the same volume V and temperature T the average number of pions in the CE with $Q = 0$ is smaller than the GCE value $3z$, thus $R_N^Q < 1$. Even stronger suppression effect is observed in the ensemble with $I = 0$, i.e. $R_N^I < R_N^Q < R_N(GCE) = 1$. The opposite effect is seen for the scaled variance of the pion number fluctuations: $\omega_N^I > \omega_N^Q > \omega_N(GCE) = 1$. To find the asymptotic behavior

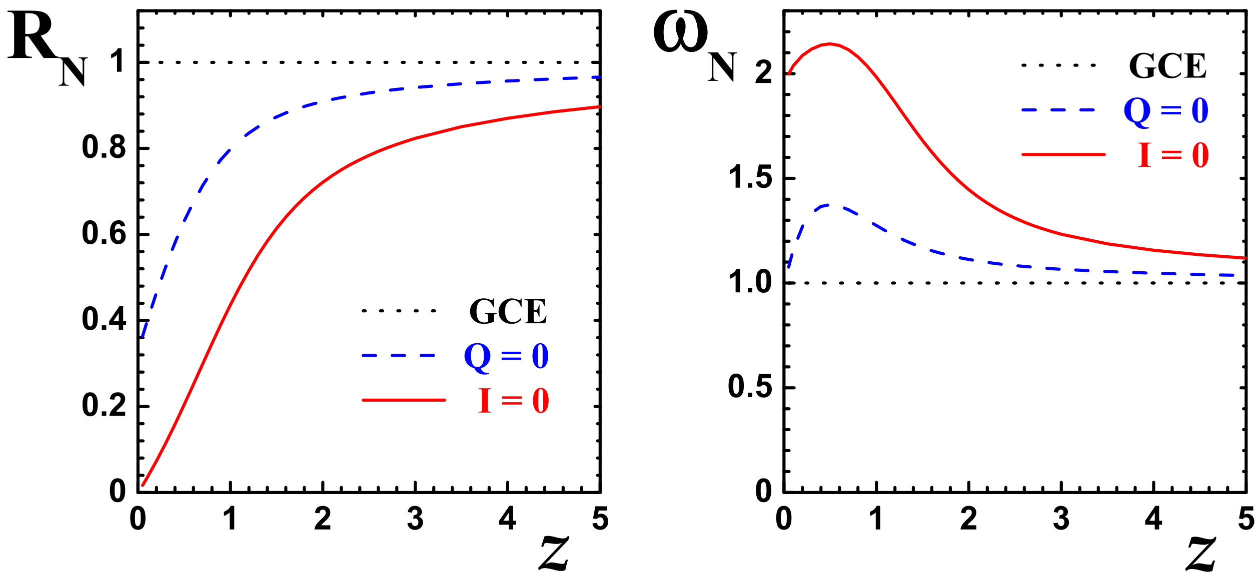

FIG. 1: The ratio R_N (*left*) and the scaled variance ω_N (*right*) in the GCE (the dotted lines), CE with Q=0 (the dashed lines), and the statistical ensemble with $I = 0$ (the solid lines)

of R_N and ω_N at large z in the statistical ensembles with $Q = 0$ and $I = 0$ one can use the expansion of the modified Bessel functions at $z \gg 1$ [13]:

$$I_n(2z) \;=\; \frac{\exp(2z)}{\sqrt{4\pi z}}\left[1 \;-\; \frac{4n^2-1}{16z} \;+\; O\left(\frac{1}{z^2}\right)\right]\;, \tag{14}$$

in Eqs. (10-12). This gives, $R_N \to 1$ and $\omega_N \to 1$ in the thermodynamic limit, $z \to \infty$, for both $Q = 0$ and $I = 0$ ensembles. It means that the suppression of the pion average multiplicity and the enhancement of the multiplicity fluctuations are the finite volume effects.

The partition functions (1), (3), and (7) can be presented as,

$$Z \;=\; \sum_{N=0}^{\infty}\sum_{N_0=0}^{N} F(N,N_0)\cdot\frac{z^N}{N!} \;=\; \sum_{N=0}^{\infty} g(N)\cdot\frac{z^N}{N!}\;. \tag{15}$$

The degeneracy factor $g(N)$ for different statistical ensembles can be found from Eq. (8). To calculate $F(N, N_0)$ one needs Eqs. (1, 3, 7). The values of $g(N)$ and $F(N, N_0)$ in different statistical ensembles are presented in Table I for $N \le 4$, see also [2]. Note that $g(N) = 3^N$ in the GCE, and $g(0) = 1$ in all ensembles. Table I is helpful when one considers very small systems with only a few pions. For $z \ll 1$ one finds from Eq. (15) and Table I:

$$Z_{Q=0} \cong 1 + 1\cdot z + 3\cdot\frac{z^2}{2!} + 7\cdot\frac{z^3}{3!} + 19\cdot\frac{z^4}{4!}\;, \qquad Z_{I=0} \cong 1 + 1\cdot\frac{z^2}{2!} + 1\cdot\frac{z^3}{3!} + 3\cdot\frac{z^4}{4!}\;. \tag{16}$$

	GCE						$Q=0$						$I=0$					
N	g(N)	$F(N,N_0)$					g(N)	$F(N,N_0)$					g(N)	$F(N,N_0)$				
	N_0	0	1	2	3	4		0	1	2	3	4		0	1	2	3	4
1	3	2	1				1	0	1				0	0	0			
2	9	4	4	1			3	2	0	1			1	2/3	0	1/3		
3	27	8	12	6	1		7	0	6	0	1		1	0	1	0	0	
4	81	16	32	24	8	1	19	6	0	12	0	1	3	6/5	0	8/5	0	1/5

TABLE I: The degeneracy factors $g(N)$ and $F(N,N_0)$ (15) in the GCE, in the CE with $Q=0$, and in the statistical ensemble with $I=0$.

The mean multiplicity and higher moments can be calculated as:

$$\langle N^k \rangle \;=\; \frac{1}{Z}\sum_{N=0}^{\infty} N^k \cdot g(N)\cdot \frac{z^N}{N!} \;\cong\; \frac{1}{Z}\sum_{N=0}^{4} N^k \cdot g(N)\cdot \frac{z^N}{N!}\;. \tag{17}$$

This gives:

$$\langle N\rangle_{Q=0} \cong\; z \;+\; 2z^2 \;-\; z^4\;, \qquad\qquad \langle N^2\rangle_{Q=0} \cong\; z \;+\; 5z^2 \;+\; 4z^3\;, \tag{18}$$

$$\langle N\rangle_{I=0} \cong\; z^2 \;+\; \frac{z^3}{2}\;, \qquad\qquad \langle N^2\rangle_{I=0} \cong\; 2z^2 \;+\; \frac{3}{2}z^3 \;+\; z^4\;. \tag{19}$$

The results (18-19) can be also obtained from Eqs. (10-12) by expanding the modified Bessel functions at $z \ll 1$ [13],

$$I_n(2z) \;=\; \frac{z^n}{n!} \;+\; \frac{z^{n+2}}{(n+1)!} \;+\; O\left(z^{n+4}\right)\;. \tag{20}$$

For R_N and ω_N (13) one finds from Eqs. (18-19):

$$R_N^Q \;\cong\; \frac{1}{3} \;+\; \frac{2}{3}\,z\;, \qquad\qquad \omega_N^Q \;\cong\; 1 \;+\; 2z \;-\; 4z^2\;, \tag{21}$$

$$R_N^I \;\cong\; \frac{1}{3}\,z\;, \qquad\qquad \omega_N^I \;\cong\; 2 \;+\frac{z}{2} \;-\; \frac{z^2}{4}\;. \tag{22}$$

The behavior of R_N (21) and ω_N (22) at small z can be seen in Fig. 1. The scaled variance ω_N has a maximum at $z \approx 0.5$ in both $Q=0$ and $I=0$ statistical ensembles.

III. FLUCTUATIONS AND CORRELATIONS OF π^0, π^+, π^-

To calculate the mean multiplicities, correlations, and fluctuations for neutral and charged pions one has to return back to presentations of the partition functions by Eqs. (1, 3, 7). One

finds,

$$\langle N_j \rangle \;=\; \frac{1}{Z}\frac{\partial Z}{\partial \lambda_j}\bigg|_{\vec{\lambda}=1} , \qquad \langle N_i\, N_j \rangle \;\equiv\; \frac{1}{Z}\frac{\partial}{\partial \lambda_i}\left(\lambda_j \frac{\partial Z}{\partial \lambda_j}\right)\bigg|_{\vec{\lambda}=1} . \tag{23}$$

Using Eq. (23) one obtains for the mean multiplicities of neutral and charged particles:

$$\langle N_0 \rangle_{gce} \;=\; \langle N_\pm \rangle_{gce} \;=\; z\;, \qquad \langle N_0 \rangle_{Q=0} \;=\; z\;, \qquad \langle N_\pm \rangle_{Q=0} \;=\; z\,\frac{I_1(2z)}{I_0(2z)} \tag{24}$$

$$\langle N_0 \rangle_{I=0} \;=\; \langle N_\pm \rangle_{I=0} \;=\; \frac{z}{3}\,\frac{I_1(2z)-I_2(2z)}{I_0(2z)-I_1(2z)}\;, \tag{25}$$

where $\langle N_\pm \rangle = \langle N_+ \rangle = \langle N_- \rangle$. The ratios

$$R_0 \;\equiv\; \frac{\langle N_0 \rangle}{z}\;, \qquad R_\pm \;=\; \frac{\langle N_\pm \rangle}{z} \tag{26}$$

are shown in Fig. 2 (*left*) for $Q = 0$ and $I = 0$ statistical ensembles.

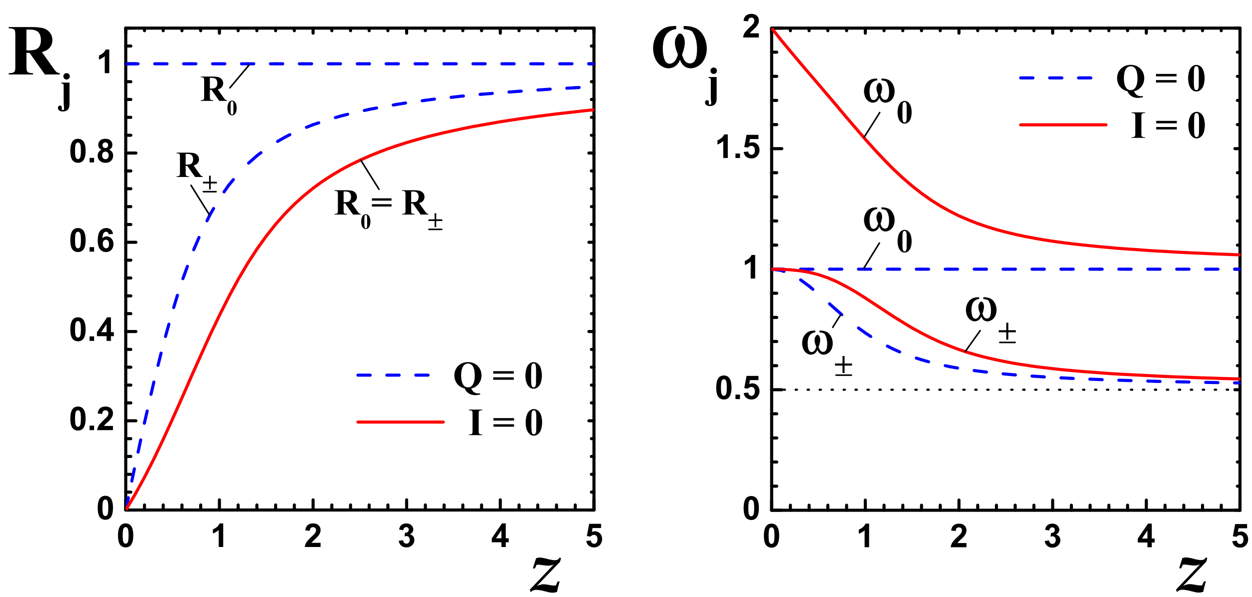

FIG. 2: The ratios R_0 and $R_\pm$ (*left*) and the scaled variances ω_0 and $\omega_\pm$ (*right*) in the CE with $Q = 0$ (the dashed lines) and in the ensemble with $I = 0$ (the solid lines).

Note that $R_0 = R_\pm = 1$ in the GCE. In the CE, $R_0^Q = 1$, i.e. the mean number of neutral pions is not affected by the charge conservation law. The mean number of charged pions is suppressed in the CE, $R_\pm^Q < 1$ [10]. The behavior in the statistical ensemble with $I = 0$ differs from that in the CE with $Q = 0$. The pion mean numbers are the same for all charge pion states π^0, π^+, and π^-. The suppression of these pion multiplicities at $I = 0$ is stronger than

that for π^- or π^+ in the CE with $Q = 0$:

$$R_0^I \;=\; R_\pm^I \;<\; R_\pm^Q \;<\; R_0^Q \;=1\;. \tag{27}$$

The asymptotic behavior of (24, 25) at large z can be found using Eq. (14). One obtains $R_j \to 1$ at $z \to \infty$ for all $j = 0, +, -$ in both $Q = 0$ and $I = 0$ statistical ensembles. Thus, the suppression of the mean numbers of charged and neutral pions is the finite volume effect. At small z one can use Eq. (20), this gives at $z \ll 1$:

$$R_\pm^Q \;\cong\; z\;, \qquad R_0^I \;=\; R_\pm^I \;\cong\; \frac{z}{3}\;. \tag{28}$$

The second derivatives in Eq. (23) can be calculated using the partition functions (1, 3, 7) for different statistical ensembles. They give the second moments and correlations in the GCE:

$$\langle N_0^2\rangle_{gce} \;=\; \langle N_\pm^2\rangle_{gce} \;=\; z \;+\; z^2\;, \qquad \langle N_0 N_\pm\rangle_{gce} \;=\; z^2\;, \tag{29}$$

in the CE:

$$\langle N_0^2\rangle_{Q=0} \;=\; z \;+\; z^2\;, \qquad \langle N_\pm^2\rangle_{Q=0} \;=\; z^2\;, \qquad \langle N_0\, N_\pm\rangle_{Q=0} \;=\; z\,\langle N_\pm\rangle_{Q=0}\;, \tag{30}$$

and in the pion system with $I = 0$:

$$\langle N_0^2\rangle_{I=0} \;=\; \langle N_0\rangle_{I=0} \;+\; \frac{1}{15}\,\frac{z\,(7z\;+\;4)\;I_0(2z)\;-\;(7z^2\;+\;2z\;+\;4)\;I_1(2z)}{I_0(2z)\;-\;I_1(2z)}\;, \tag{31}$$

$$\langle N_\pm^2\rangle_{I=0} \;=\; \langle N_\pm\rangle_{I=0} \;+\; \frac{z^2}{5}\,\frac{I_2(2z)\;-\;I_3(2z)}{I_0(2z)\;-\;I_1(2z)}\;, \tag{32}$$

$$\langle N_0\, N_\pm\rangle_{I=0} \;=\; \frac{z}{15}\,\frac{z\,I_1(2z)\;-\;(z-3)\,I_2(2z)}{I_0(2z)\;-\;I_1(2z)}\;, \tag{33}$$

where $\langle N_\pm^2\rangle = \langle N_+^2\rangle = \langle N_-^2\rangle = \langle N_+ N_-\rangle$.

Using the above expressions one can easily construct the scaled variances ω_j and correlation coefficients ρ_{ij},

$$\omega_j \;\equiv\; \frac{\langle N_j^2\rangle\;-\;\langle N_j\rangle^2}{\langle N_j\rangle}\;, \qquad \rho_{ij} \;\equiv\; \frac{\langle N_i N_j\rangle\;-\;\langle N_i\rangle\langle N_j\rangle}{\sqrt{\omega_i\,\omega_j\,\langle N_i\rangle\,\langle N_j\rangle}}\;, \tag{34}$$

that define the main characteristics of the pion multiplicity distributions.

From Eqs. (24-25, 31-32) and asymptotic expansion of the modified Bessel functions (14) one finds the behavior of the scaled variances ω_j (34) in the thermodynamic limit $z \to \infty$. As seen from Fig. 2, the isospin conservation with $I = 0$ gives the same pion number fluctuations as the CE with $Q = 0$, $\omega_0^I \to 1$ and $\omega_\pm^I \to 1/2$, at $z \to \infty$.

However, the results for finite systems are rather different. In the CE with $Q = 0$ the fluctuations of neutral particles are the same as in the GCE, $\omega_0^Q = 1$. The scaled variance $\omega_{\pm}^Q$ in the CE with $Q = 0$ was calculated in Ref. [9]. At $z \ll 1$ using Eq. (20), one finds: $\omega_{\pm}^Q \cong 1 - z^2/2$.

In the statistical ensemble with $I = 0$ the scaled variances ω_0 and $\omega_{\pm}$ can be calculated for small system with $z < 1$ using Eqs. (24-25, 31-32) and asymptotic expansion of the modified Bessel functions (20). However, it is more instructive to obtain approximate expressions for the distributions of π^0 and $\pi^{\pm}$ numbers using Eq. (15) and $F(N, N_0)$ values from Table I. Let us consider the pion states with total number of pions $N \leq 4$, i.e. we neglect small terms of the order of z^5. The probability distribution $P_0(N_0)$ then reads:

$$P_0(N_0) \;\cong\; \frac{1}{Z}\sum_{N=0}^{4} F(N, N_0)\,\frac{z^N}{N!}\;. \tag{35}$$

For the statistical ensemble with $I = 0$ one finds from Eq. (35) and Table I,

$$P_0(1) \;\cong\; \frac{1}{Z_{I=0}}\;1\cdot\frac{z^3}{3!} \;\cong\; \frac{z^3}{6}\;, \tag{36}$$

$$P_0(2) \;\cong\; \frac{1}{Z_{I=0}}\left[\frac{1}{3}\cdot\frac{z^2}{2!} \;+\; \frac{8}{5}\cdot\frac{z^4}{4!}\right] \cong\; \frac{z^2}{6} \;-\; \frac{z^4}{60}\;, \tag{37}$$

$$P_0(3) \;\cong\; 0\;, \qquad P_0(4) \;\cong\; \frac{1}{Z_{I=0}}\;\frac{1}{5}\cdot\frac{z^4}{4!} \;\cong\; \frac{z^4}{120}\;. \tag{38}$$

Similar expressions for $P_{\pm}(N_{\pm})$ in the statistical ensemble with $I = 0$ are equal to:

$$P_{\pm}(1) \;\cong\; \frac{1}{Z_{I=0}}\left[\frac{2}{3}\cdot\frac{z^2}{2!} \;+\; 1\cdot\frac{z^3}{3!} \;+\; \frac{8}{5}\cdot\frac{z^4}{4!}\right] \;\cong\; \frac{z^2}{3} \;+\; \frac{z^3}{6} \;-\; \frac{z^4}{10}\;, \tag{39}$$

$$P_{\pm}(2) \;\cong\; \frac{1}{Z_{I=0}}\left[\frac{6}{5}\cdot\frac{z^4}{4!}\right] \;\cong\; \frac{z^4}{20}\;, \qquad P_{\pm}(3) \;\cong\; P_{\pm}(4) \;\cong\; 0\;. \tag{40}$$

The distributions P_0 and $P_{\pm}$ are rather different. The one and two particle states have evidently different probabilities for neutral and (negative) positive pions. At $z \ll 1$ the main configurations consist of 1 (negative) positive pion and 2 neutral pions. These states correspond to the total number of pions $N = 2$, and their probabilities are proportional to z^2. The states with only 1 neutral pion have smaller probability, z^3, as they can only appear for $N \geq 3$. Inspite of differnces in the P_0 and $P_{\pm}$ distributions they give, however, the same average numbers of neutral and (negative) positive pions:

$$\langle N_0\rangle_{I=0} \cong 1\cdot P_0(1) \;+\; 2\cdot P_0(2) \;+\; 4\cdot P_0(4) \;\cong\; \frac{z^2}{3} \;+\; \frac{z^3}{6}\;, \tag{41}$$

$$\langle N_{\pm}\rangle_{I=0} \cong 1\cdot P_{\pm}(1) \;+\; 2\cdot P_{\pm}(2) \cong\; \frac{z^2}{3} \;+\; \frac{z^3}{6}\;. \tag{42}$$

Equations (41,42) clearly demonstrate that the same average numbers of π^0 and $\pi^\pm$ come from different pion states. The results for higher moments of P_0 and $P_\pm$ distributions are different:

$$\langle N_0^2\rangle_{I=0} \cong 1^2 \cdot P_0(1) \ + \ 2^2 \cdot P_0(2) \ + \ 4^2 \cdot P_0(4) \cong \frac{2z^2}{3} \ + \ \frac{z^3}{6} \ + \ \frac{z^4}{15}~, \tag{43}$$

$$\langle N_\pm^2\rangle_{I=0} \cong 1^2 \cdot P_\pm(1) \ + \ 2^2 \cdot P_\pm(2) \ \cong \ \frac{z^2}{3} \ + \ \frac{z^3}{6} \ + \ \frac{z^4}{10}~. \tag{44}$$

From Eqs. (41-44) one finds:

$$\omega_0^I \ \cong \ 2 \ - \ \frac{z}{2}~, \quad \omega_\pm^I \ \cong \ 1 \ - \ \frac{z^2}{30}~. \tag{45}$$

There are no correlations between the numbers of π^0, π^+, π^- in the GCE. All correlation coefficients defined by Eq. (34) are equal to zero, $\rho_{0+} = \rho_{0-} = \rho_{+-} = 0$. The charge is exactly conserved in the $Q = 0$ and $I = 0$ statistical ensembles. This brings the strongest correlations between the numbers of π^+ and π^-, i.e. $\rho_{+-} = 1$, and this means equal numbers N_+ and N_- in each microscopic state of the system. The correlations between the numbers of π^0 and $\pi^\pm$, are absent in the CE with $Q = 0$, but exist in the statistical ensemble with $I = 0$. The correlation coefficient $\rho_{0\pm}^I$ at $I = 0$ is presented in Fig. 3. It has the maximal value $\rho_{0\pm}^I \approx 0.19$ at $z \approx 1$ and goes to zero at $z \to \infty$.

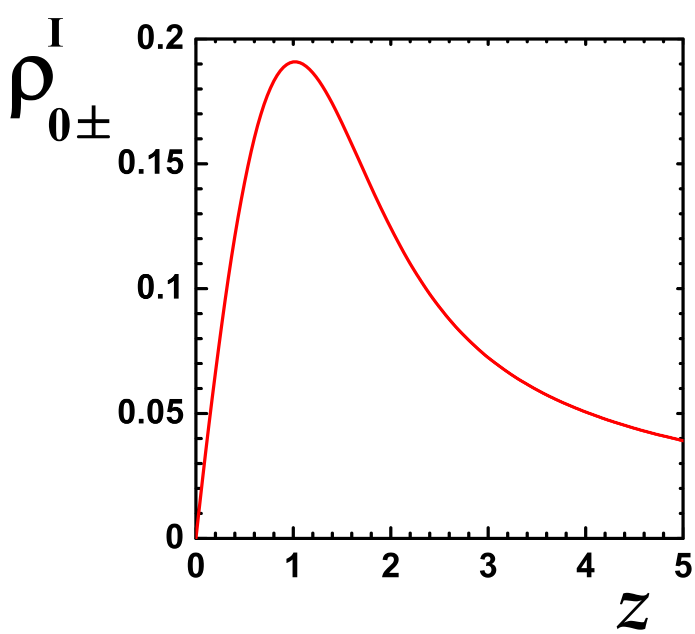

FIG. 3: The correlation coefficient $\rho_{0\pm}^I$ (34) between the neutral and charged pions in the statistical ensemble with $I = 0$.

The role of the pion number correlations can be illustrated returning to the scaled variance ω_N presented in Fig. 1, *right*. Taking into account the relation $N \equiv N_0 + N_+ + N_-$ one finds:

$$\omega_N \;=\; \omega_0\,\frac{\langle N_0\rangle}{\langle N\rangle} \;+\; 2\omega_\pm\,\frac{\langle N_\pm\rangle}{\langle N\rangle} \;+\; 2\rho_{+-}\,\omega_\pm\,\frac{\langle N_\pm\rangle}{\langle N\rangle} \;+\; 2\rho_{0\pm}\,\sqrt{\omega_0\omega_\pm}\,\frac{\langle N_\pm\rangle}{\langle N\rangle}\;. \qquad (46)$$

The GCE corresponds to $\rho_{+-} = \rho_{0\pm} = 0$ in Eq. (46). Besides, the multiplicities $\langle N_j\rangle = \langle N\rangle/3$ and scaled variances $\omega_j = 1$ are the same for $j = 0, +, -$. This leads to $\omega_N = 1$ in the GCE.

The CE with $Q = 0$ corresponds to $\omega_0^Q = 1$, $\rho_{+-}^Q = 1$ and $\rho_{0\pm}^Q = 0$ in Eq. (46). This gives,

$$\omega_N^Q \;=\; \frac{\langle N_0\rangle_{Q=0}}{\langle N\rangle_{Q=0}} \;+\; 4\omega_\pm^Q\,\frac{\langle N_\pm\rangle_{Q=0}}{\langle N\rangle_{Q=0}} \;\cong\; (1-2z+4z^2) \;+\; 4(z-2z^2) \;=\; 1+2z-4z^2\;. \quad (47)$$

The term $\langle N_0\rangle_{Q=0}/\langle N\rangle_{Q=0}$ in Eq. (47) decreases at small z. The maximum of ω_N^Q appears due to $\rho_{+-}^Q = 1$, note the factor 4 in the second term in r.h.s. of Eq. (47). Due to the exact charge conservation, negative and positive pions may appear only as $\pi^+\pi^+$-pairs. For $N_{ch} \equiv N_+ + N_-$, the scaled variance

$$\omega_{ch}^Q \;=\; \frac{\langle N_{ch}^2\rangle_{Q=0} \;-\; \langle N_{ch}\rangle_{Q=0}^2}{\langle N_{ch}\rangle_{Q=0}} \;=\; 2\;\omega_-^Q \;=\; 2\;\omega_+^Q \qquad (48)$$

is then two times larger than that for positive (negative) pions. The same relation, $\omega_{ch} = 2\omega_\pm$, is also valid at $I = 0$.

For $I = 0$ one obtains from Eq. (46):

$$\begin{aligned}\omega_N^I \;&=\; \frac{1}{3}\left(\omega_0^I \;+\; 4\omega_\pm^I \;+\; 4\rho_{0\pm}^I\,\sqrt{\omega_0^I\omega_\pm^I}\right)\\ &\cong\; \left(\frac{2}{3}-\frac{z}{6}+\frac{7z^2}{180}\right) \;+\; \left(\frac{4}{3}-\frac{2z^2}{45}\right) \;+\; \left(\frac{2z}{3}-\frac{11z^2}{45}\right) \;=\; 2 \;+\; \frac{z}{2} \;-\; \frac{z^2}{4}\;. \end{aligned} \qquad (49)$$

where the relation $\langle N_0\rangle_{I=0} = \langle N_\pm\rangle_{I=0} = \langle N\rangle_{I=0}/3$ is used. The sum of $\omega_0^I/3$ and $4\omega_\pm^I/3$ decreases at small z. An increase of ω_N^I and its maximum at small z comes due to the third term in the r.h.s. of Eq. (49), i.e. due to the correlations between the neutral and charged pions, $\rho_{0\pm}^I > 0$. Note that a conclusive comparison with experimental data for small pion systems would require an inclusion of additional conservation laws like 4-momentum, angular momentum, etc. [15].

IV. BOSE STATISTIC

In this section we discuss the role of Bose effects in the statistical ensemble with fixed isopsin. The role of Bose effects for the fluctuations in the CE was considered in Ref. [14]. A

generalization of Eq. (6) for Bose statistics gives the following expression:

$$Z_{I=0}^{Bose} = \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta \tag{50}$$
$$\times \ \exp\left[\sum_{n=1}^{\infty} \frac{z_n}{n} \left(\lambda_0^n \cos^n\beta \ + \ \left(\frac{1+\cos\beta}{2}\right)^n \left(\lambda_+^n \exp[in(\alpha+\gamma)] \ + \ \lambda_-^n \exp[-in(\alpha+\gamma)]\right)\right)\right] ,$$

where

$$z_n = \frac{V}{2\pi^2} \frac{Tm^2}{n} K_2\left(\frac{nm}{T}\right) . \tag{51}$$

The Boltzmann approximation corresponds to the first term $n = 1$ in the sum in Eq. (50). The results for ω_j^I are presented in Fig. 4.

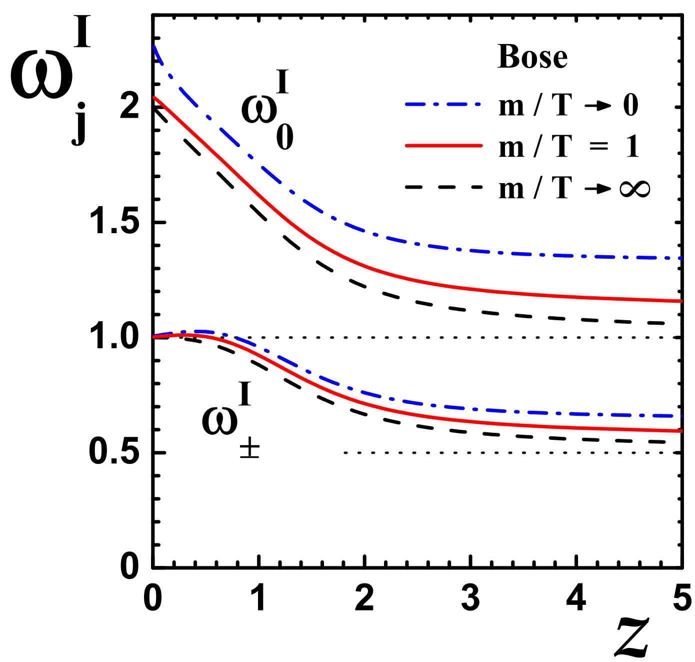

FIG. 4: The scaled variances ω_0^I and $\omega_\pm^I$ in the Bose gas with $I = 0$. The solid lines correspond to $m/T = 1$, dashed-dotted lines to $m/T \to 0$, and dashed lines to $m/T \to \infty$.

In the case of quantum statistics, the partition function depends not only on the one particle partition function z (2), but additionally on the value of m/T. In Fig. 4 the results for the scaled variances ω_j^I are shown for a typical value $m/T = 1$ and for two limiting cases: $m/T \to 0$ when the Bose effects are the strongest ones, and $m/T \to \infty$ when the Bose effects disappear and results coincide with the Boltzmann approximation presented in Fig. 2, *right*. As seen from Fig. 4, Bose statistics makes the pion number fluctuations larger. These effects are always

stronger for smaller values of the m/T ratio. For $m/T \to 0$ we find $\omega_0^I \cong 2.26$ at $z \to 0$, and $\omega_0^I \cong 1.368$ at $z \to \infty$. The corresponding results for the charged particles are the following: $\omega_\pm^I = 1$ at $z \to 0$, and $\omega_\pm^I \cong 0.684$ at $z \to \infty$. The results for $z \to \infty$ coincide with those in the canonical ensemble with $Q = 0$.

The Bose statistics effects become visible when more than one identical pion (i.e. pions in the same charge states) can appear in the same quantum state. Note a difference between the neutral and charged pions for very small systems with $I = 0$. At $z << 1$ the main configurations consist of 1 (negative) positive pion and 2 neutral pions. These states correspond to the total number of pions $N = 2$ and their probabilities are proportional to z^2. The states with only 1 neutral pion have smaller probability, z^3, as they can only appear for $N \geq 3$. Because of this difference one observes no Bose effects for $\omega_\pm^I$ at $z \to 0$ in Fig. 4, whereas these effects are seen for ω_0^I.

V. SUMMARY

Particle number fluctuations and correlations in the statistical system of pions with zero total isospin $I = 0$ have been studied in the present paper. For finite systems one observes a suppression of the average total pion number and an increase of the pion number fluctuations. The suppression effects due to isospin conservation are the same for average numbers of π^0, π^+ and π^-. However, we find quite different behavior of the corresponding scaled variances. For neutral pions there is the enhancement of the fluctuations, whereas for charged pions the isospin conservation suppresses fluctuations similar to that in the canonical ensemble with $Q = 0$. The positive correlations between the numbers of neutral and (negative) positive pions are observed for $I = 0$. This effects is absent in the canonical ensemble with $Q = 0$. The correlations between the numbers of neutral and charged pions are responsible for a maximum of ω_N for small systems with $I = 0$. In the thermodynamic limit the correlation coefficient $\rho_{0\pm}^I$ between neutral and charged pions goes to zero. This leads to $\omega_0^I \to 1$, $\omega_\pm^I \to 1/2$ for Boltzmann statistics and to $\omega_0^I \to 1.368$, $\omega_\pm^I \to \omega_0^I/2 = 0.684$ for $m = 0$ and Bose statistics.

Finally, we speculate on possible consequences of our results for the quark-gluon gas with SU(3)-color group. A requirement of colorless of the system of quarks and gluons is similar to that of the isospin singlet $I = 0$. This requirement cause the suppression effects for the average yields of quarks and gluons, the same for different colors. These suppression effects

for the yields disappear in the thermodynamic limit. However, a behavior of the fluctuations are different. To make situation fully transparent let us consider a toy model of the gluon gas with SU(2)-color. The colorless system of gluons will be then in one-to-one correspondence with the iso-singlet $I = 0$ pion gas. We thus immediately conclude that fluctuations of the number of gluons with different colors are different. Moreover, this difference survives in the thermodynamic limit too.

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